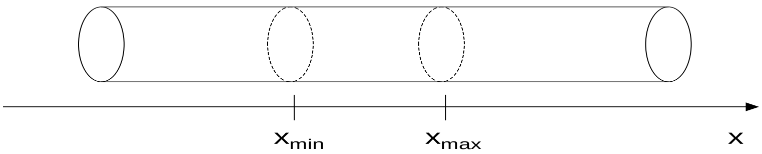
UNIVERSITY OF BOLOGNA - FACULTY OF ENGINEERING

INTERNATIONAL MASTER COURSE IN CIVIL ENGINEERING - A.Y. 2012/2013

INTRODUCTION TO NUMERICAL METHODS

LAB 6: ONE-SPACE DIMENSIONAL ADVECTION PDE

Let us consider a substance (e.g. a liquid or a gas) flowing in a tube-shaped region of space having constant cross section A. Let $u(x,t)$ denote the density of the substance (mass $\times$ volume^{-1}), $v(x,t)$ the velocity of the substance (length $\times$ time^{-1}) and $\phi(x,t)$ the flux of substance (mass $\times$ time$^{-1}\times$ area^{-1}) across the cross section. All these quantities are assumed to be functions of time t and of a unique one-dimensional space variable x, that is density, velocity and flux variations in the y and z directions are assumed to be negligible.



The amount (mass) of substance contained at a generic time t in a generic finite subset (that is, in a generic interval $x_{min} \le x \le x_{max}$) of the tube-shaped region is:

$$M(t) = \int_{x_{min}}^{x_{max}} u(x,t)\, A dx = A \int_{x_{min}}^{x_{max}} u(x,t)\, dx$$

The net flux of substance $F(t)$ (mass $\times$ time^{-1}) entering the interval at time t is:

$$F(t) = \phi(x_{min},t)\, A - \phi(x_{max},t)\, A = A\left(\phi(x_{min},t) - \phi(x_{max},t)\right)$$

The law of mass conservation implies for the rate of increase of the total mass of substance:

$$\frac{d}{dt}(M(t)) = F(t) \;\Rightarrow\; \frac{d}{dt}\Big(A \int_{x_{min}}^{x_{max}} u(x,t)\, dx\Big) = A(\varphi(x_{min},t) - \varphi(x_{max},t)) \;\Rightarrow\; \int_{x_{min}}^{x_{max}} (u_t + \varphi_x)\, dx = 0$$

Since the interval $[x_{min}, x_{max}]$ is arbitrary, the conservation equation should hold at every point:

$$u_t + \phi_x = 0 \quad \forall (x,t)$$

But we know (it is easy to demonstrate) that:

$$\phi(x,t) = v(x,t) \cdot u(x,t)$$

so that the conservation equation can be written as:

$$u_t + (v\,u)_x = 0$$

If we assume that the substance velocity is constant in time and space, that is $v(x,t) = a \in R \;\; \forall (x,t)$, we obtain:

$$u_t + a\,u_x = 0$$

This is the one-space dimensional transport (advection) PDE. Given the velocity a and an initial (for $t = 0$) profile $u(x,t=0) = u_0(x)$ of the substance density in the one-dimensional tube-shaped region, we can calculate all the subsequent (for $t > 0$) density profiles by solving the following problem:

$$\begin{array}{ll} (1a) & \\ (1b) & \end{array} \begin{cases} u_t + a\,u_x \;\; = \;\; 0 & -\infty < x < +\infty\,, \quad t > 0 \\ u(x,t=0) \; = \; u_0(x) & -\infty < x < +\infty \end{cases} \qquad (1)$$

where (1a) is the (one-space dimensional, homogeneous) advection PDE with $a \in R$ the PDE parameter (velocity), (1b) is the initial condition with $u_0 : R \mapsto R$ a scalar-valued function of the one-dimensional space variable, (1) = (1a) + (1b) is called an initial-value problem: given the values of the unknown function $u(x,t)$ at an initial time $t = 0$, that is, for example, an initial measure of the state of a modelled system (density of the substance in our example), we are interested in determining how the system evolves in time, that is in computing the values of $u(x,t)$ for $t > 0$.

It is easy to derive the analytical solution of the problem (1):

$$u(x,t) = u_0(x - at) \qquad (2)$$

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a)

Consider the initial-value differential problem (1), namely the one-space dimensional advection PDE:

$$\begin{cases} u_t + a\,u_x \;=\; 0 & -\infty < x < +\infty\,, \quad t > 0 \\ u(x, t=0) \;=\; u_0(x) & -\infty < x < +\infty \end{cases}$$

where $a \in R$ and $u_0 : R \mapsto R$ are the problem data, that is the advection PDE parameter (velocity) and the initial condition function, respectively.

As reported in (2), this problem admits a close-form analytical solution:

$$u_e(x,t) = u_0(x - at)$$

In MATLAB, create a script that computes numerically an approximate solution of the above problem by using the finite difference method:

After setting the data of the problem, that is the velocity a and the initial condition function $u_0(x)$ (directly in the script or asking to the user), and the space and time discretization steps h and k (so that the space-time domain mesh for the finite difference method is univocally defined), the script must compute the approximate values of the solution at the mesh nodes inside a properly chosen finite space domain [x_{min}, x_{max}]. At each time step t_j, the script must show in a Figure the exact solution $u_e(x,t_j)$ and the computed approximate solution $u_e(x_i,t_j)$.

As far as the finite difference method is concerned, implement and test for different data (velocity and initial condition) and different space-time discretization steps (h and k) the two explicit schemes FTBS (Forward-Time, Backward-Space) and FTFS (Forward-Time, Forward-Space). What about the accuracy of the solution? What about stability? Discuss the results.

If you want, implement also the two implicit schemes BTBS (Backward-Time, Backward-Space), BTFS (Backward-Time, Forward-Space) and discuss the results.